PHY 456 Lecture 5:

Spin-$\frac{1}{2}$ Rotations, Pauli Spinors

Continuation of Lecture 4.

How do we implement our knowledge of SU(2), SO(3) groups and complex pairs $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ into QM?

We had: $[\hat{L}_a, \hat{L}_b] = i\hbar \epsilon_{abc} \hat{L}_c$

and $\sigma_a \sigma_b = 2\delta_{ab} + i\epsilon_{abc}\sigma_c$

$\Rightarrow [\sigma_a, \sigma_b] = 2i\,\epsilon_{abc}\sigma_c$

$\Rightarrow [\frac{\hbar}{2}\sigma_a, \frac{\hbar}{2}\sigma_b] = i\hbar\,\epsilon_{abc}\frac{\hbar}{2}\sigma_c$

Look!

which is identical to the angular momentum commutator!

Define: $\hat{S}_a \equiv \frac{\hbar}{2}\sigma_a$ Operator for spin for spin-$\frac{1}{2}$ particles

THEN $\hat{L}_a, \hat{S}_a$ obey the same Lie Algebra.

What are the eigenvalues of $\hat{S}_a$?

Since $\hat{S}_a$ and $\hat{\vec{S}}_a^2$ don't commute $[\hat{S}_a, \hat{\vec{S}}_a^2]=0$,

we can simultaneously diagonalize $\hat{S}_z$ and $\hat{\vec{S}}^2$:

$$\hat{\vec{S}}^2 = \sum_{a=1}^{3} \frac{\hbar}{2}\sigma_a \frac{\hbar}{2}\sigma_b \delta_{ab}$$

$$= \frac{\hbar^2}{4}\sum_{a=1}^{3} \sigma_a^2$$

$$= \frac{3}{4}\hbar^2 \mathbb{1}$$

but notice that $\frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right)$ just like $\ell(\ell+1)$ for $\vec{L}^2$...

The 2-dimensional Hilbert Space for spin-$\frac{1}{2}$ is

$$\left.\begin{matrix} \left| s=\frac{1}{2},\ s_z=\frac{1}{2} \right\rangle \\ \left| s=\frac{1}{2},\ s_z=-\frac{1}{2} \right\rangle \end{matrix}\right\} \text{2-dimensional, like } \{|\ell, m\rangle\ ;\ m=-\ell,\dots,\ell\}$$

which we can denote as 'basis vectors'

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv \left|\frac{1}{2},\frac{1}{2}\right\rangle \qquad \left\{\left|\ell=\frac{1}{2}, m\right\rangle\ ;\ m=-\frac{1}{2},\frac{1}{2}\right\}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \equiv \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Since $\hat{\vec{S}}^2 = \hbar^2\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\mathbb{1}$, both are eigenvectors of $\hat{\vec{S}}^2$.

Using $\hat{S}_z$, we find that

$$\hat{S}_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\hat{S}_z\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

which looks okay.

We can extend this to a general rotation operator for spin states:

$$\underbrace{{}^{(\frac{1}{2})}}_{\text{spin}}\hat{R}(\underbrace{\alpha}_{\text{angle}}, \overset{\text{axis}}{\vec{n}}) = e^{-\frac{i}{\hbar}\alpha\vec{n}\cdot\hat{\vec{S}}}$$

spin $\frac{1}{2}$ operator

while

$$\underbrace{{}^{(r)}}_{\text{orbital angular momentum}}\hat{R}(\overset{\text{angle}}{\alpha}, \overset{\text{axis}}{\vec{n}}) = e^{-\frac{i}{\hbar}\alpha\vec{n}\cdot\hat{\vec{L}}}$$

$\hat{L}_i = \epsilon_{ijk}\hat{r}_j\hat{p}_k$

$\therefore \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

$= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

Similar to our classical rotation operator $^{(r)}R(\alpha, \hat{z}) = e^{\alpha(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})}$

The $^{(\frac{1}{2})}\hat{R}(\alpha, \vec{n})$ operator acts in the $\begin{pmatrix}z_1\\z_2\end{pmatrix} = z_1\begin{pmatrix}1\\0\end{pmatrix} + z_2\begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{C}^2$ states, but we can also write it using σ- matrices instead:

$$^{(\frac{1}{2})}\hat{R}(\alpha, \vec{n}) = e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}}$$

$$\equiv u$$

previously called u!

which is a 2×2 unitary matrix with unit determinant.

Properties:

1) $$\det\left(e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}}\right) = e^{\mathrm{tr}\left(-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}\right)}$$

$$= e^{-i\frac{\alpha}{2}\vec{n}\cdot\mathrm{tr}(\vec{\sigma})}$$

$$= 1$$

Proof (see tutorial):

$$e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = \cos\frac{\alpha}{2} - i\,\vec{n}\cdot\vec{\sigma}\,\sin\frac{\alpha}{2}$$

which is unitary.

2) $$e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}}\,\sigma_a\,e^{i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = \tilde{R}_{ab}(\alpha,\vec{n})\,\sigma_b$$

$SO(3)$ rotation matrix...

$$u\sigma_b u^{+} = \tilde{R}_{bc}\,\sigma_c$$

Proof: See HW/tutorial.

$$e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}}\,\sigma_a\,e^{i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = \left(\cos\frac{\alpha}{2} - i\,\vec{n}\cdot\vec{\sigma}\sin\frac{\alpha}{2}\right)\sigma_a\left(\cos\frac{\alpha}{2} + i\,\vec{n}\cdot\vec{\sigma}\sin\frac{\alpha}{2}\right)$$

$$= \left(\cos\frac{\alpha}{2}\sigma_a - i\,\vec{n}\cdot\vec{\sigma}\,\sigma_a\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2} + i\,\vec{n}\cdot\vec{\sigma}\sin\frac{\alpha}{2}\right)$$

$$= \cos^2\frac{\alpha}{2}\sigma_a - i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(\vec{n}\cdot\vec{\sigma}\,\sigma_a - \sigma_a\,\vec{n}\cdot\vec{\sigma}\right) + \vec{n}\cdot\vec{\sigma}\,\sigma_a\,\vec{n}\cdot\vec{\sigma}\,\sin^2\frac{\alpha}{2} \quad [1]$$

Now $$\vec{n}\cdot\vec{\sigma}\,\sigma_a - \sigma_a\,\vec{n}\cdot\vec{\sigma} = \sum_b n_b(\sigma_b\sigma_a - \sigma_a\sigma_b)$$

$$= \sum_b n_b[\sigma_b,\sigma_a]$$

$$= 2i\sum_b n_b\,\epsilon_{bac}\,\sigma_c$$

Furthermore, using $\{\sigma_a, \sigma_c\} = 2\delta_{ac}$, observe that

$$\vec{n}\cdot\vec{\sigma}\,\sigma_a = -\sigma_a\,\vec{\sigma}\cdot\vec{n} + 2n_a,$$

hence

$$\vec{n}\cdot\vec{\sigma}\,\sigma_a\,\vec{n}\cdot\vec{\sigma} = -\sigma_a(\vec{n}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) + 2n_a\,\vec{n}\cdot\vec{\sigma}$$

$$= -\sigma_a + 2n_a\,\vec{n}\cdot\vec{\sigma}.$$

Back into equation [1], we find

$$= \cos^2\tfrac{\alpha}{2}\,\sigma_a + 2\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\sum_b n_b\,\epsilon_{bac}\,\sigma_c$$

$$+ 2\sin^2\tfrac{\alpha}{2}\,n_a\,\vec{n}\cdot\vec{\sigma} - \sin^2\tfrac{\alpha}{2}\,\sigma_a$$

$$= \underbrace{\left(\cos^2\tfrac{\alpha}{2} - \sin^2\tfrac{\alpha}{2}\right)}_{\cos\alpha}\sigma_a - \sin\alpha\,\epsilon_{abc}\,n_b\,\sigma_c$$

$$+ \underbrace{2\sin^2\tfrac{\alpha}{2}}_{(1-\cos\alpha)}\,n_a\,\vec{n}\cdot\vec{\sigma}$$

$$= \cos\alpha\,\sigma_a + (1-\cos\alpha)\sum_c n_a n_c\sigma_c - \sum_{b,c}\sin\alpha\,\epsilon_{abc}\,n_b\,\sigma_c$$

$$= \underbrace{\left(\cos\alpha\,\delta_{ac} + (1-\cos\alpha)\,n_a n_c - \sin\alpha\,\epsilon_{abc}\,n_b\right)}_{R_{ac}}\sigma_c$$

Thus $R\sigma_a R^{\dagger} = \sum_c R_{ac}\,\sigma_c$;

$$\boxed{R_{ac}(\vec{n},\alpha) = \cos\alpha\,\delta_{ac} + (1-\cos\alpha)\,n_a n_c - \sin\alpha\,\epsilon_{abc}\,n_b.}$$

We can check this for rotations around the z axis:

$n = \delta^{a3}$, and we find

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Determinant is obviously 1.

Notice that $^{(\frac{1}{2})}R(\alpha, \vec{n}) = e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}}$ allows us to rotate spin states!

Complex Spin-$\frac{1}{2}$ States

Let us discuss how to describe a particle with spin-$\frac{1}{2}$ (ie, electron).

The Hibert space is that of so-called Pauli Spinors;

$$[\psi](\vec{r}) = \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

ψ_+: $S_z = +\frac{\hbar}{2}$

ψ_-: $S_z = -\frac{\hbar}{2}$

are wavefunctions of electron spins

and also $[\varphi](\vec{r}) = \begin{pmatrix} \varphi_+(\vec{r}) \\ \varphi_-(\vec{r}) \end{pmatrix}$ be another spinor.

Then

$$\langle\varphi|\psi\rangle = \int d^3r \begin{pmatrix} \varphi_+^*(\vec{r}) & \varphi_-^*(\vec{r}) \end{pmatrix} \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

inner product of 2 Pauli Spinors

$$= \int d^3r \left(| \psi_+^*(\vec{r}) \psi_+(\vec{r}) | + | \psi_-^*(\vec{r}) \psi_-(\vec{r}) | \right)$$

For a single $[\psi]$, we normalize:

$$1 = \langle \psi | \psi \rangle = \int d^3r \left(|\psi_+(\vec{r})|^2 + |\psi_-(\vec{r})|^2 \right)$$

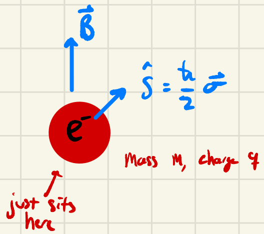
which implies that $|\psi_+(\vec{r})|^2 \, d^3\vec{r}$ = probability to find e^- with spin "up" $S_z = +\hbar/2$ in $(\vec{r}, \vec{r} + d^3\vec{r})$

$|\psi_-(\vec{r})|^2 \, d^3\vec{r}$ = probability to find e^- with spin "down" $S_z = -\hbar/2$ in $(\vec{r}, \vec{r} + d^3\vec{r})$

and the total probability must sum to 1.

Consider the following simple spin-related problem of a particle in a magnetic field $\vec{B}$:

- e^- particle fixed in space
- $\vec{B}$ is constant



Then $\hat{H} = -\frac{q}{m} \vec{B} \cdot \hat{\vec{S}}$ acts on $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ Hilbert space. Note $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ is not $\vec{r}$-dependent (and $\vec{B}$ is constant).

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = -\frac{q}{m} \vec{B} \cdot \hat{\vec{S}} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

Taking $\vec{B} = B\hat{z}$, no loss of generality,

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = -\frac{q}{m}\frac{\hbar}{2} B\sigma_3 \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}.$$

Let $\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \equiv \psi_+\begin{pmatrix}1\\ 0\end{pmatrix} + \psi_-\begin{pmatrix}0\\ 1\end{pmatrix}$, then

in σ_3, S_z basis

$$\begin{pmatrix} i\hbar\,\dot{\psi}_+ \\ i\hbar\,\dot{\psi}_- \end{pmatrix} = -\frac{q\hbar}{2m} B \begin{pmatrix}\psi_+\\ -\psi_-\end{pmatrix}$$

and we obtain

$$i\hbar\,\dot{\psi}_\pm = \mp \frac{q\hbar}{2m}\psi_\pm$$

$$\Longrightarrow \boxed{\psi_\pm(t) = e^{\pm i\frac{qBt}{2m}}\psi_\pm(0)}$$

Consider the initial state $\psi_\pm(0) = \begin{pmatrix}\psi_+(0)\\ \psi_-(0)\end{pmatrix}$. If $\psi_-(0)=0$, $\psi_+(0) = 1$

hence $\begin{pmatrix}\psi_+(t)\\ \psi_-(t)\end{pmatrix} = e^{i\frac{qBt}{2m}}\begin{pmatrix}1\\ 0\end{pmatrix}$ with $\hat{\vec{S}}$ is $\parallel$ $\vec{B}$ only a phase,

<u>BUT</u> if $\psi_+(0) = \psi_-(0) = \frac{1}{\sqrt{2}}$, say, then spin precesses!

See tutorial.